Semigroups of Möbius transformations

Matthew Jacques

Thursday 12th March 2015

- Joint work with Ian Short -
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1 Möbius transformations and hyperbolic geometry
   ▶ Möbius transformations and their action inside the unit ball
   ▶ The hyperbolic metric

2 Semigroups of Möbius transformations
   ▶ Semigroups
   ▶ Limit sets of Möbius semigroups
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3 Composition sequences
   ▶ Escaping and converging composition sequences
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4 A Theorem on convergence
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   - Möbius transformations and their action inside the unit ball
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4 A Theorem on convergence
Möbius transformations are the *conformal automorphisms* of \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \).

That is, the bijective functions on \( \hat{\mathbb{C}} \) which preserve angles and their orientation.

Each takes the form

\[
    z \mapsto \frac{az + b}{cz + d}
\]

with \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \).
Möbius transformations

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We consider the group $\mathcal{M}$ of Möbius transformations acting on $\hat{\mathbb{C}}$, which we identify with $\mathbb{S}^2$.

By decomposing the action of any given Möbius transformation into a composition of inversions in spheres orthogonal to $\mathbb{S}^2$, the action of $\mathcal{M}$ may be extended to a conformal action on $\mathbb{R}^3 \cup \{\infty\}$.

In particular $\mathcal{M}$ gives a conformal action on the closed unit ball, which it preserves.

This extension is called the Poincaré extension.
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The hyperbolic metric, $\rho(\cdot , \cdot)$ on $\mathbb{B}^3$

The hyperbolic metric $\rho$ on $\mathbb{B}^3$ is induced by the infinitesimal metric

$$ds = \frac{|dx|}{1 - |x|^2}.$$  

- From any point inside $\mathbb{B}^3$ the distance to the ideal boundary, $\mathbb{S}^2$, is infinite.
- Geodesics are circular arcs which when extended land orthogonally on $\mathbb{S}^2$.

The group of Möbius transformations that preserve $\mathbb{B}^3$ are exactly the set of orientation preserving isometries of $(\mathbb{B}^3, \rho)$.
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Möbius transformations and hyperbolic geometry

The hyperbolic metric

Matthew Jacques (The Open University)

Semigroups of Möbius transformations

Thursday 12\textsuperscript{th} March 2015
Möbius transformations and hyperbolic geometry

The hyperbolic metric
Aside from the identity, there are three types of Möbius transformation.

- **Loxodromic transformations**
  Conjugate to $z \mapsto \lambda z$ where $|\lambda| \neq 1$.
  Have two fixed points, one attracting and one repelling.

- **Elliptic transformations**
  Conjugate to $z \mapsto \lambda z$ where $|\lambda| = 1$.
  Have two neutral fixed points.

- **Parabolic transformations**
  Conjugate to $z \mapsto z + 1$.
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Semigroups

**Definition**

Given a set $\mathcal{F}$ of Möbius transformations, the *semigroup* $S$ generated by $\mathcal{F}$ is the set of finite (and non-empty) compositions of elements from $\mathcal{F}$.

We write $S = \langle \mathcal{F} \rangle$ as the semigroup generated by $\mathcal{F}$.
Limit sets

Let $S$ be a semigroup of Möbius transformations.

**Definition**

The forwards limit set of $S$ is the set

$$\Lambda^+(S) = \left\{ z \in \mathbb{S}^2 \mid \lim_{n \to \infty} g_n(\zeta) = z \text{ for some sequence } g_n \text{ in } S \right\}.$$ 

Similarly the backwards limit set of $S$ is given by

$$\Lambda^-(S) = \left\{ z \in \mathbb{S}^2 \mid \lim_{n \to \infty} g_n^{-1}(\zeta) = z \text{ for some sequence } g_n \text{ in } S \right\}.$$ 

Since each $g_n$ is an isometry of the hyperbolic metric, these definitions are independent of the choice of $\zeta \in \mathbb{B}^3$. 
Three characterisations

Write

\[ J(S) = \text{subset of } \mathbb{S}^2 \text{ upon which } S \text{ is not a normal family}. \]

**Theorem**  D. Fried, S. Marotta and R. Stankewitz (2012)

For except for certain ”Elementary” semigroups,

\[
\begin{align*}
\Lambda^-(S) &= J(S) &= \{\text{Repelling fixed points of } S\} \\
\Lambda^+(S) &= J(S^{-1}) &= \{\text{Attracting fixed points of } S\}.
\end{align*}
\]
Properties  (Fried, Marotta and Stankewitz)

- Both $\Lambda^+$, $\Lambda^-$ are closed.

- Either $|\Lambda^+| < 3$ or $\Lambda^+$ is a perfect set. Similarly for $\Lambda^-$. 

- $\Lambda^+$ is *forward invariant under* $S$, that is $g(\Lambda^+) \subseteq \Lambda^+$ for all $g \in S$.

- If $\Lambda^+$ contains at least two points then it is the smallest closed forwards invariant set containing at least two points.

- $\Lambda^-$ is *backwards invariant under* $S$, that is $g^{-1}(\Lambda^-) \subseteq \Lambda^-$ for all $g \in S$.

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Elementary semigroups

\[ \mathcal{F} = \left\{ z \mapsto e^{i\theta} z \right\} \]
\[ \Lambda^- = \Lambda^+ = \emptyset. \]

\[ \mathcal{F} = \{ z \mapsto 2z \} \]
\[ \Lambda^- = \{ 0 \}, \quad \Lambda^+ = \{ \infty \}. \]

\[ \mathcal{F} = \{ z \mapsto z + 1 \} \]
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\[ \mathcal{F} = \left\{ z \mapsto \frac{1}{3} z, \quad z \mapsto \frac{1}{3} z + \frac{2}{3} \right\} \]
\[ \Lambda^- = \{ \infty \}, \quad \Lambda^+ = \text{middle thirds Cantor set.} \]
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Non-elementary Kleinian group

A Kleinian group is a group $S$ such that the $S$ orbit of any point in hyperbolic space is a discrete set of points.

Any Kleinian group is a semigroup with equal forwards and backwards limit sets.
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Subsemigroup of a Kleinian Group

Consider the Modular group \( \Gamma \).

\( \Gamma \) may be generated by two parabolic generators, \( f, g \).
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Consider the Modular group $\Gamma$. 
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\[
\begin{array}{c}
\circlearrowleft \quad f \\
\circlearrowright \quad g
\end{array}
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Subsemigroup of a Kleinian Group

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![Diagram showing $f^{-1}$, $f$, $g^{-1}$, and $g$.]
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Consider the Modular group $\Gamma$. $\Gamma$ may be generated by two parabolic generators, $f, g$.

Let $S$ be the semigroup generated by $f, g$. 
Subsemigroup of a Kleinian Group

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Examples of non-elementary semigroups
Semigroups of Möbius transformations

Examples of non-elementary semigroups

Matthew Jacques (The Open University)
Schottky Semigroups
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\[ S = \langle \{ f, g, h, h^{-1} \} \rangle \]
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\[ S = \langle \{f, g, h, h^{-1}\} \rangle \]
Fix a set of Möbius transformations $\mathcal{F}$.

A composition sequence of Möbius transformations generated by $\mathcal{F}$ is any sequence with $n^{\text{th}}$ term

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n,$$

where each $f_i$ is chosen from $\mathcal{F}$.

Note the direction of composition.
Composition sequences

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Note the direction of composition.
Write \( f(z) = \frac{1}{z + 2} \) and \( g(z) = \frac{3}{z + 1} \). Then

\[
F_1(z) = f(z) = \frac{1}{z + 2}
\]

\[
F_2(z) = f \circ g(z) = \frac{1}{\frac{3}{2 + \frac{1}{1 + z}}}
\]

\[
F_3(z) = f \circ g \circ g(z) = \frac{1}{\frac{3}{2 + \frac{3}{1 + \frac{1}{1 + z}}}}
\]

so that \( F_n(0) \) is the \( n^{th} \) convergent of some continued fraction.
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so that $F_n(0)$ is the $n^{th}$ convergent of some continued fraction.
Escaping sequences

Definition

We say a sequence of Möbius transformations \( g_n \) is escaping if \( g_n \zeta \) accumulates only on the boundary of hyperbolic space. Equivalently

\[
\rho(g_n \zeta, \zeta) \to \infty \quad \text{as} \quad n \to \infty.
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Converging sequences

**Definition**

We say a sequence $g_n$ converges if $g_n \zeta$ accumulates at exactly one point on the boundary of hyperbolic space.
Converging sequences

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Composition sequences

Escaping and converging sequences

- \( \mathcal{F} = \{ z \mapsto \frac{1}{3} z, \ z \mapsto \frac{1}{3} z + \frac{2}{3} \} \)
  
  Every composition sequence escapes? \( \checkmark \)
  
  Every composition sequence converges? \( \checkmark \)

- \( \mathcal{F} \) such that \( \mathcal{F} \) generates a group.
  
  Every composition sequence escapes? \( \times \)
  
  Every composition sequence converges? \( \times \)
Composition sequences
Escaping and converging sequences

- \( \mathcal{F} = \left\{ z \mapsto \frac{1}{3}z, \ z \mapsto \frac{1}{3}z + \frac{2}{3} \right\} \)

Every composition sequence escapes?  ✔
Every composition sequence converges?  ✔

- \( \mathcal{F} \) such that \( \mathcal{F} \) generates a group.

Every composition sequence escapes?  ✗
Every composition sequence converges?  ✗
Composition sequences

Escaping and converging sequences

• \( F = \left\{ z \mapsto \frac{1}{3}z, \ z \mapsto \frac{1}{3}z + \frac{2}{3} \right\} \)

Every composition sequence escapes? ✓

Every composition sequence converges? ✓

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Composition sequences

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- \( F \) such that \( F \) generates a group.
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Composition sequences  Escaping and converging sequences

- $\mathcal{F} = \left\{ z \mapsto \frac{1}{3} z, \ z \mapsto \frac{1}{3} z + \frac{2}{3} \right\}$

  Every composition sequence escapes? ✓
  Every composition sequence converges? ✓

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  Every composition sequence converges? ✓

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  Every composition sequence escapes?  ✔
  Every composition sequence converges?  ✔

- $\mathcal{F}$ such that $\mathcal{F}$ generates a group.
  Every composition sequence escapes?  ❌
  Every composition sequence converges?  ❌
**Question:**
Given a particular composition sequence, does it converge?

**Related question:**
Given a set of Möbius transformations $\mathcal{F}$ when does every composition sequence generated by $\mathcal{F}$

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- converge?
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Given a set of Möbius transformations $\mathcal{F}$ when does every composition sequence generated by $\mathcal{F}$

- escape,
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Let $S = \langle F \rangle$ be the semigroup generated by $F$.

**Proposition**

Every composition sequence generated by $F$ escapes if and only if $\text{Id} \notin \overline{S}$.

**Proposition**

If $\Lambda^+$ and $\Lambda^-$ are disjoint then every escaping composition sequence generated by $F$ converges.

On the other hand:

**Proposition**

There is a dense $G_\delta$ set (w.r.t. the topology on $\Lambda^-$), $D^-$ contained in $\Lambda^-$ such that if $\Lambda^+$ meets $D^-$, then not every composition sequence generated by $F$ converges.
Let $S = \langle \mathcal{F} \rangle$ be the semigroup generated by $\mathcal{F}$.

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**Theorem**

Suppose $\mathcal{F}$ is a bounded set of Möbius transformations acting on $\mathbb{B}^2$, generating a non-elementary semigroup $S$.

Every composition sequence drawn from $\mathcal{F}$ converges if and only if $\text{Id} \notin \overline{S}$ and $\Lambda^+$ is not the whole of $\mathbb{S}^1$. 
Lemma

If $S$ is a semigroup of Möbius transformations acting on $\mathbb{B}^3$ such that $|\Lambda^-| > 1$ and if $\Lambda^- \subseteq \Lambda^+$, then there exists a composition sequence in $S$ that does not converge.

Whenever $\Lambda^+ = S^1$ there exists some composition sequence that does not converge.
Lemma

If $S$ is a semigroup of Möbius transformations acting on $\mathbb{B}^3$ such that $|\Lambda^-| > 1$ and if $\Lambda^- \subseteq \Lambda^+$, then there exists a composition sequence in $S$ that does not converge.

Whenever $\Lambda^+ = \mathbb{S}^1$ there exists some composition sequence that does not converge.
**Optimal?**

Can we drop the reference to $\Lambda^+ = S^1$, in other words is the following true?

**Conjecture**

Suppose $\mathcal{F}$ is bounded set of Möbius transformations acting on $\mathbb{B}^2$, generating a non-elementary semigroup $S$. Every composition sequence drawn from $\mathcal{F}$ converges if and only if every composition sequence escapes.
Literature

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Thank you for your attention!