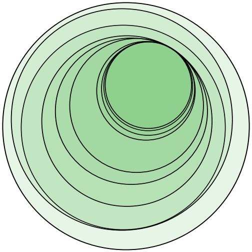
Composition sequences of Möbius maps



Matthew Jacques

Monday 1st September 2014

- Joint work with Ian Short -

We are interested in *composition sequences* of Möbius transformations

$$F_n = g_1 \circ g_2 \circ \cdots \circ g_n.$$

We fix a Euclidean disc D and choose a finite set of Möbius transformations $\mathcal{F}$, that each maps D into, but not onto itself.

We then consider composition sequences

$$F_n = g_1 \circ g_2 \circ \cdots \circ g_n,$$

where each g_i is chosen from $\mathcal{F}$.

For such composition sequences $F_n(D)$ is a nested sequence of discs.

$$g_n(D) \subset D$$
$$F_n(D) \subset F_{n-1}(D).$$

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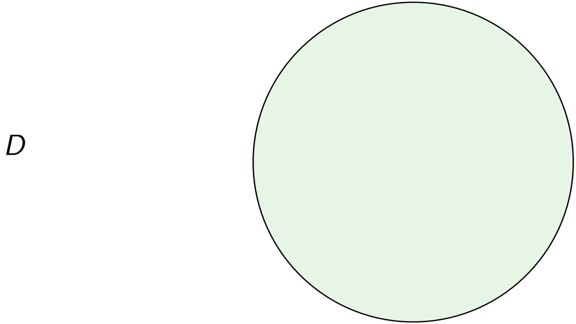

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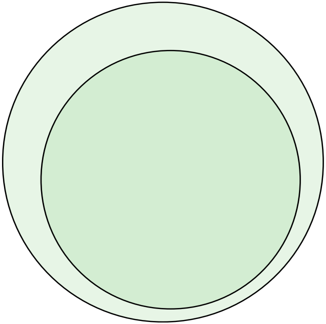
$F_1(D)$



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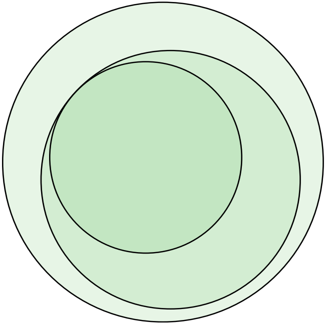
$F_2(D)$



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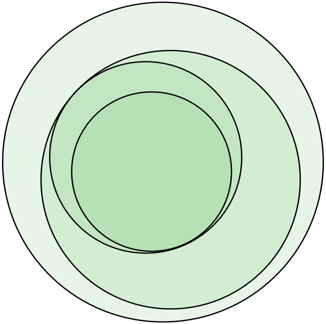
$F_3(D)$



For such composition sequences $F_n(D)$ is a nested sequence of discs.

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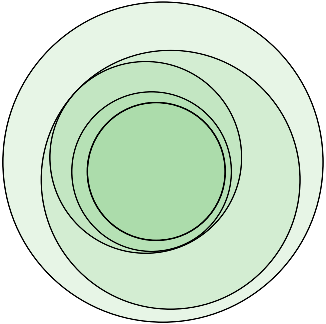
$F_4(D)$



So there are two possibilities, either

- $\bigcap_n F_n(\overline{D})$ is a point or,
- $\bigcap_n F_n(\overline{D})$ is a disc.

This talk is concerned with studying this dichotomy.

We call the former case *limit-point case*, while the latter we call *limit-disc case*.

Theorem (Hillam, Thron 1965)

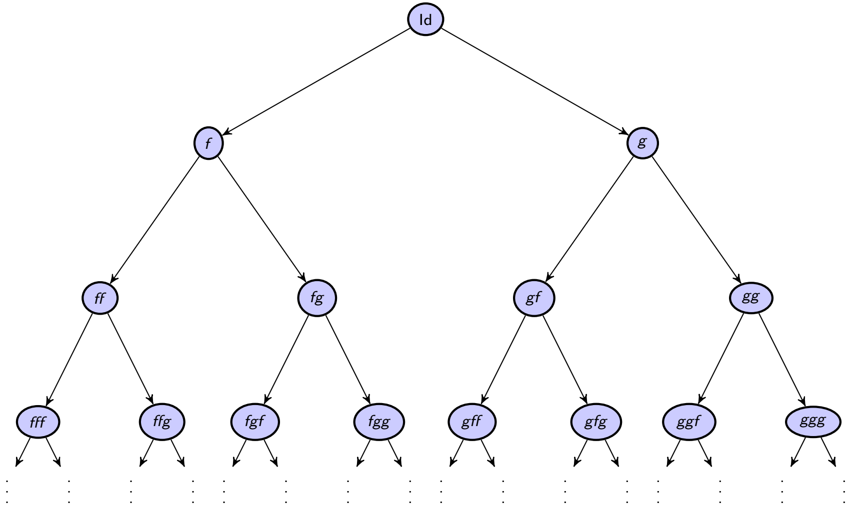
Suppose D is a disc and that $\mathcal{F}$ is a finite set of Möbius transformations which each map D into, but not onto itself.
Further suppose

$$F_n = g_1 \circ g_2 \circ \cdots \circ g_n,$$

is a composition sequence such that each g_i is drawn from $\mathcal{F}$.
Then F_n converges locally uniformly in D to a constant function.

As a consequence of this theorem, limit-point and limit-disc case are geometric descriptions of uniform, and local uniform convergence respectively.

Another way to think about composition sequences, is as infinite branches in the tree of words generated by $\mathcal{F}$. So for example if $\mathcal{F} = \{f, g\}$.



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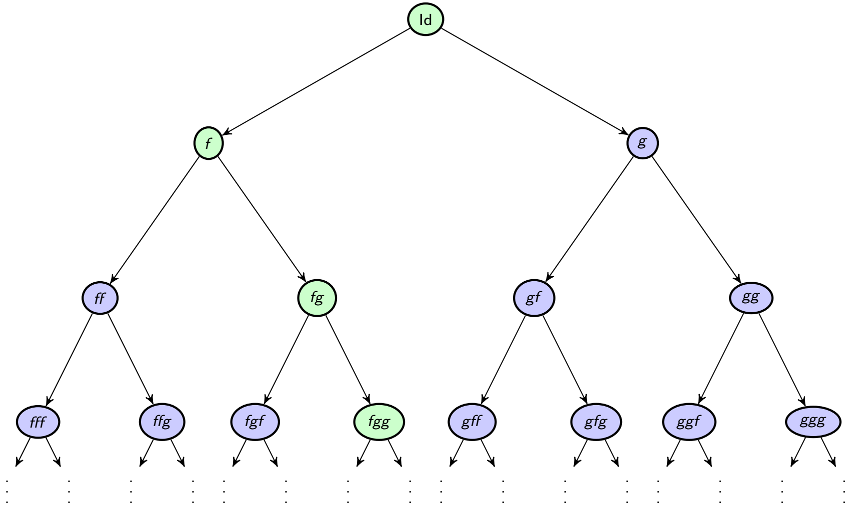
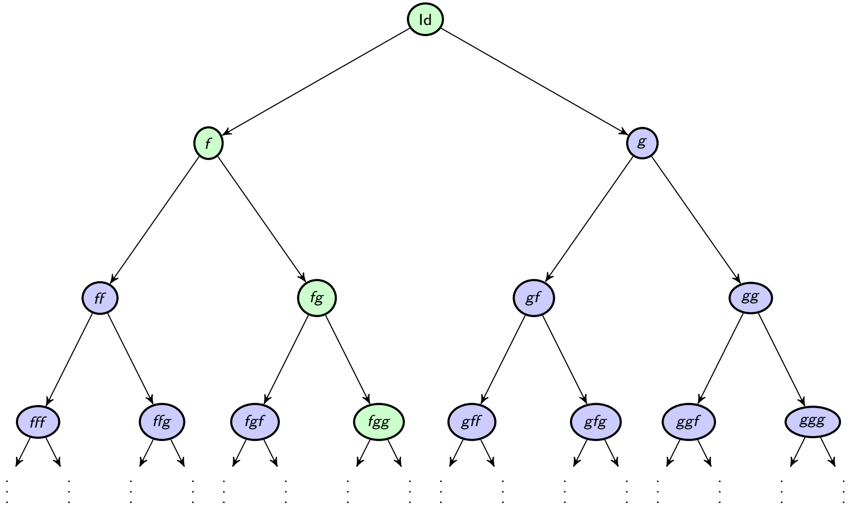


Another way to think about composition sequences, is as infinite branches in the tree of words generated by $\mathcal{F}$. So for example if $\mathcal{F} = \{f, g\}$.
The vertices in the tree are the elements of the semigroup generated by $\mathcal{F}$ under composition.



A consequence of limit-disc case

Theorem (J, Short)

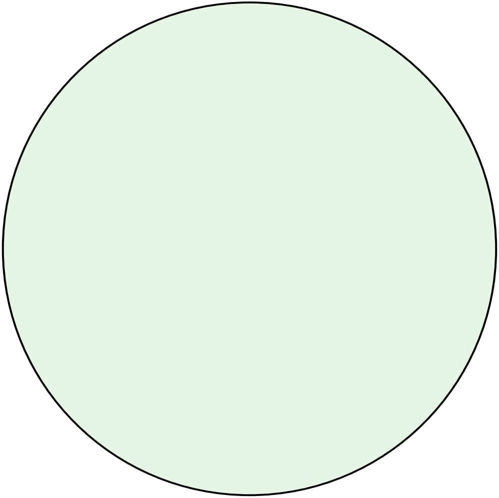
Suppose D is a disc and that $\mathcal{F}$ is a finite set of Möbius transformations which each map D into, but not onto itself.
Let

$$F_n = g_1 \circ g_2 \circ \cdots \circ g_n.$$

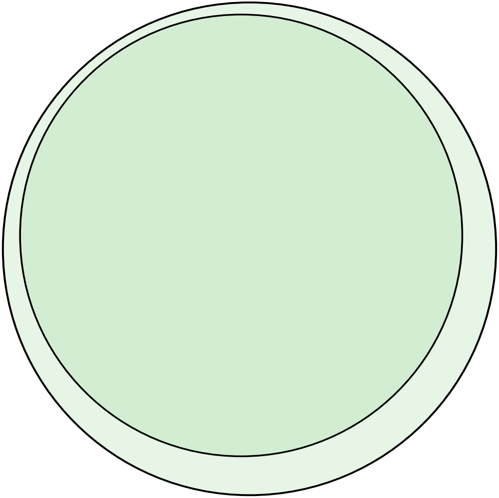
be a composition sequence such that each g_i is chosen from $\mathcal{F}$.

If F_n is of limit-disc type then the sequence of nested discs $F_n(D)$ are eventually mutually tangent with a common point of tangency.

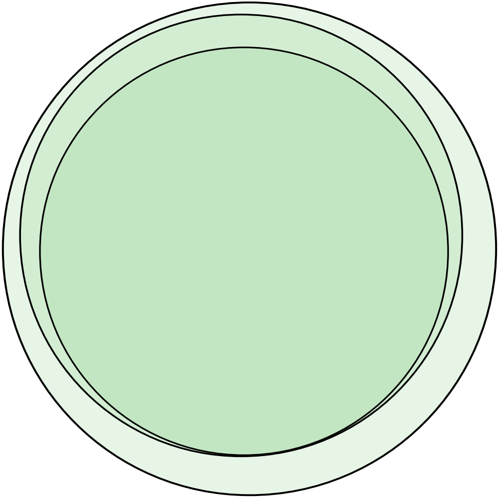
The nested sequence of discs $F_n(D)$: D



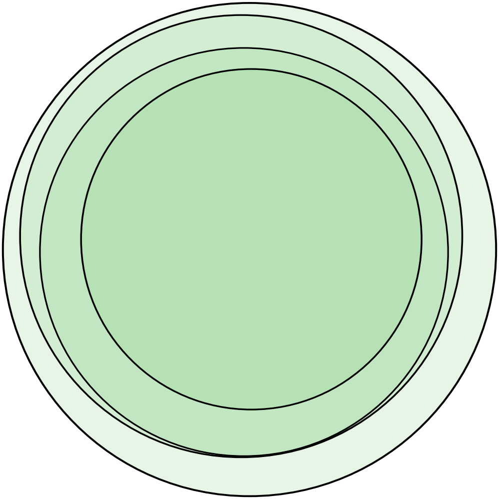
The nested sequence of discs $F_n(D)$: $F_1(D)$



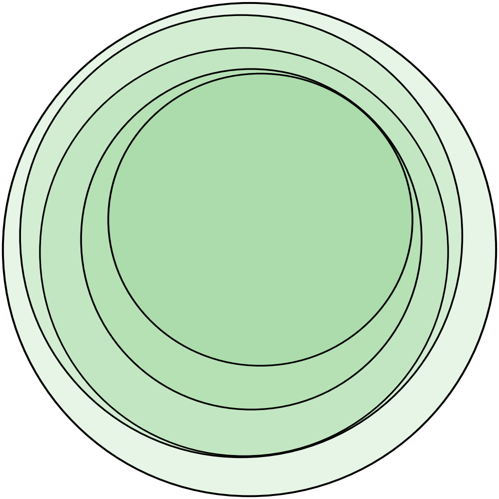
The nested sequence of discs $F_n(D)$: $F_2(D)$



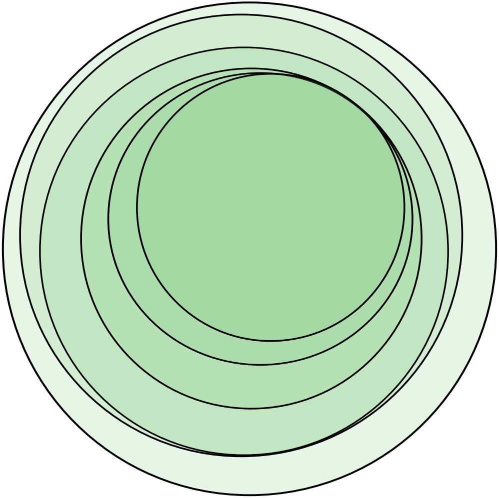
The nested sequence of discs $F_n(D)$: $F_3(D)$



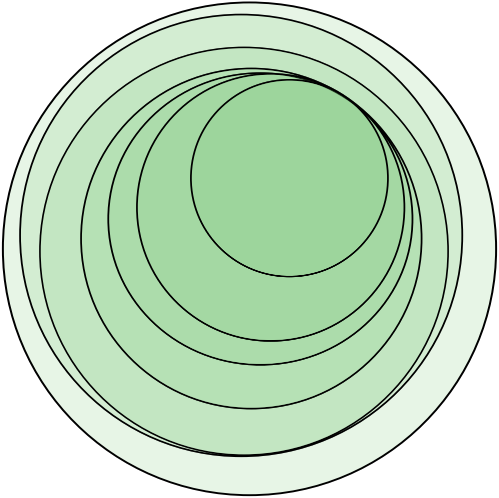
The nested sequence of discs $F_n(D)$: $F_4(D)$



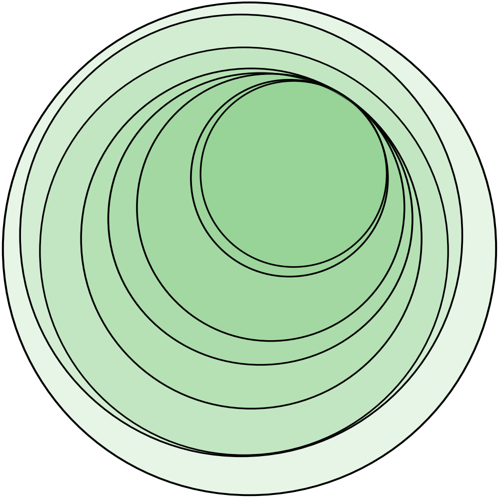
The nested sequence of discs $F_n(D)$: $F_5(D)$



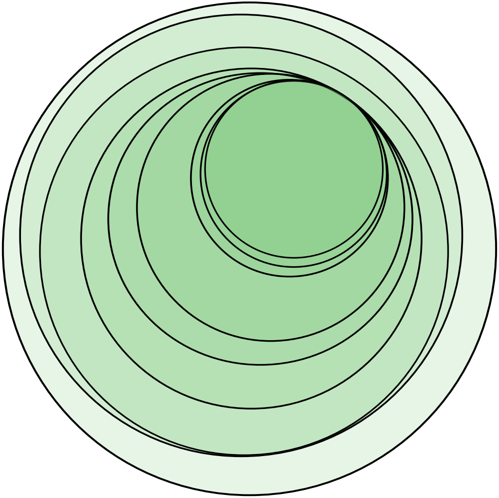
The nested sequence of discs $F_n(D)$: $F_6(D)$



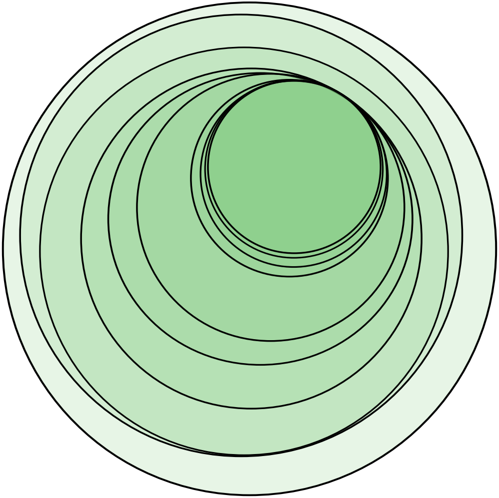
The nested sequence of discs $F_n(D)$: $F_7(D)$



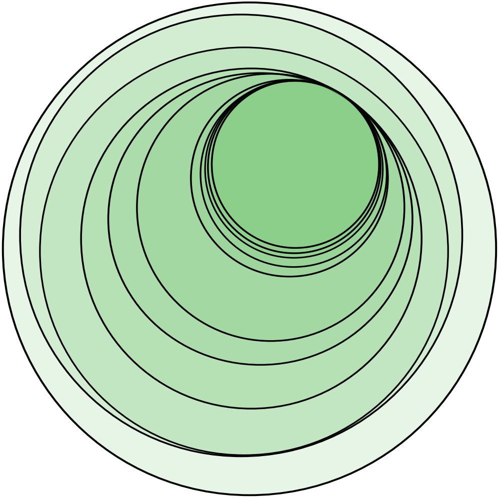
The nested sequence of discs $F_n(D)$: $F_8(D)$



The nested sequence of discs $F_n(D)$: $F_9(D)$



The nested sequence of discs $F_n(D)$: $F_n(D)$



Theorem (J, Short)

Suppose D is a disc and that $\mathcal{F}$ is a finite set of Möbius transformations which each map D into, but not onto itself.

Let

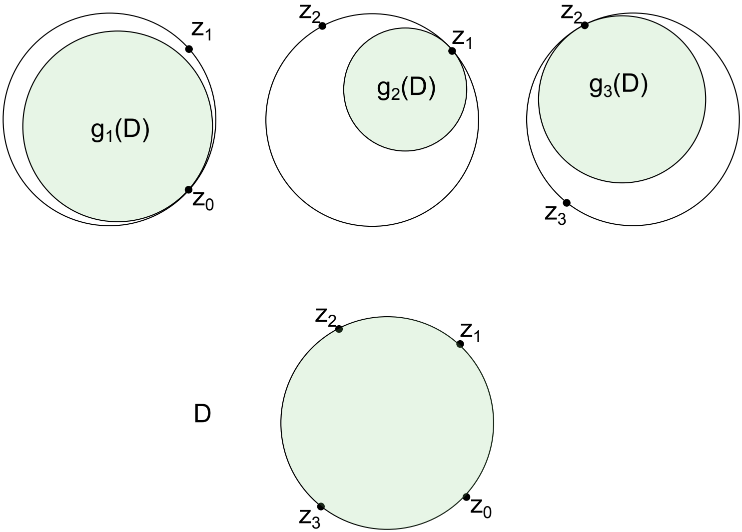
$$F_n = g_1 \circ g_2 \circ \cdots \circ g_n.$$

be a composition sequence such that each g_i is chosen from $\mathcal{F}$.

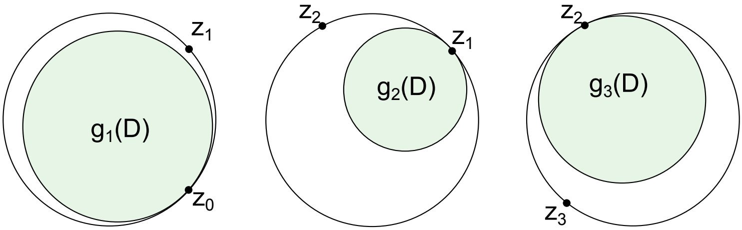
Then F_n is limit-disc case if and only if:

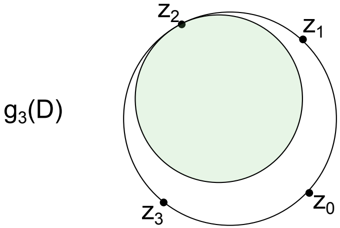
(i) There is a sequence z_n on ∂D such that for all big enough n, $g_n(z_n) = z_{n-1}$.

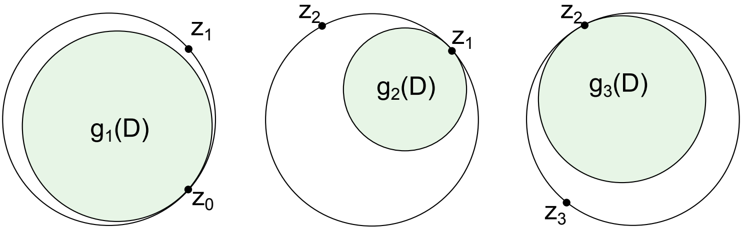
(ii) $\sum_{n=1}^{\infty} 1/|g_1^{'}(z_1)| \dots |g_n^{'}(z_n)| < +\infty$.



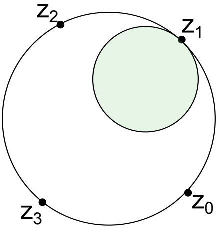
z_1
$g_1(D)$
z_0
z_2
z_1
$g_2(D)$
z_2
$g_3(D)$
z_3
z_2
z_1
D
z_3
z_0

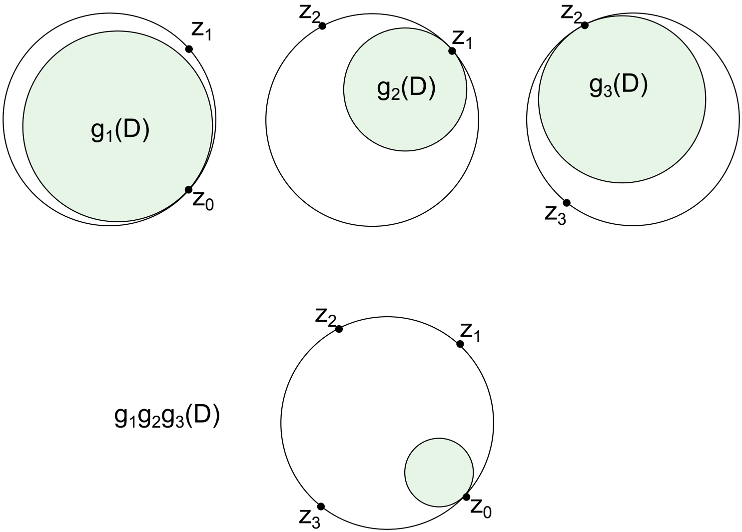

z1
z0
g1(D)
z2
z1
g2(D)
z2
g3(D)
z3


g3(D)
z2
z1
z3
z0



$g_2g_3(D)$





z_1
$g_1(D)$
z_0
z_2
z_1
$g_2(D)$
z_2
$g_3(D)$
z_3
$g_1g_2g_3(D)$
z_2
z_1
z_3
z_0

How common is limit-disc case?

The preceding Theorem says limit-disc case requires a very special set of circumstances:

- Firstly our pool of transformations $\mathcal{F}$ must contain functions that map D to a disc tangent to D.
- Every such map sends exactly one point on the boundary of D to another point on the boundary of D, and these points must (eventually) somehow line up.
- Finally, not too many of the g_n may be too contracting.

Towards quantifying how many sequences are limit-disc case

First choose an enumeration of $\mathcal{F}$, say $\mathcal{F} = \{f_0, f_1, \dots f_{b-1}\}$.
Let $\mathcal{S}$ denote the set of sequences taking values in $\{0, \dots, b-1\}$.
To any $x \in \mathcal{S}$ we associate it with the composition sequence that has the following n^{th} term

$$f_{x(1)} \circ f_{x(2)} \circ \cdots \circ f_{x(n)}$$

and to the real number with base b expansion

$$0.x(1)x(2)\dots x(n)\dots$$

We define the Hausdorff dimension of any $X \subseteq \mathcal{S}$, $\dim(X)$, to be the usual Hausdorff dimension of its associated set contained in $[0, 1]$.

For example, if $\mathcal{F} = \{f_0, f_1, f_2\}$ then the set of sequences

$$\{x \in \mathcal{S} \mid (x(0), x(1), x(2)) = (2, 0, 1)\}$$

is associated with the set of real numbers

$$[0.201\overline{0}, 0.201\overline{2}]$$

(base 3) which has Hausdorff dimension 1.

Let $\mathcal{S}_\circ$ denote the subset of $\mathcal{S}$ which corresponds to composition sequences of limit-disc case.

Question

Given a finite set $\mathcal{F}$, what is the Hausdorff dimension of $\mathcal{S}_\circ$?

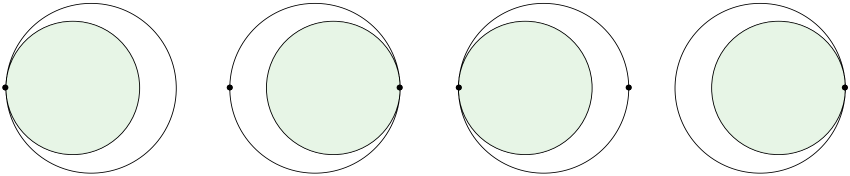
Towards answering this question, we introduce the following idea:

Definition

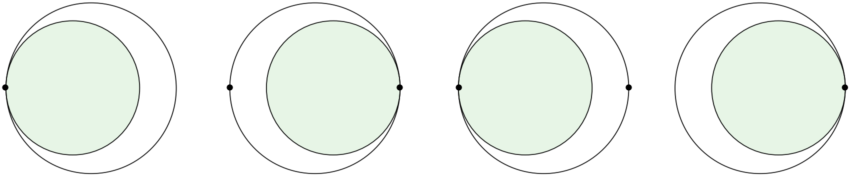
The *tangency graph* G of $\mathcal{F}$ is a directed graph whose vertices are the set $\mathcal{F}$.
Given vertices f, g (not necessarily distinct) there is a directed edge from f to g exactly when $f \circ g(D)$ is tangent to D.

For example choosing D to be the $R = \frac{1}{2}(3 + \sqrt{5})$ radius disc centered at the origin, let $\mathcal{F} = \{f_0, f_1, f_2, f_3\}$ where each f_i is as below.

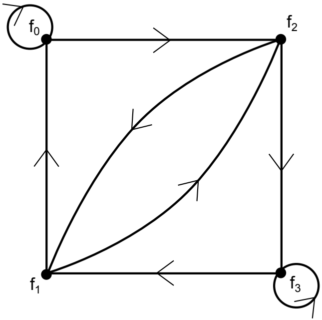
$$f_0(z) = \frac{-1}{z+3} \qquad f_1(z) = \frac{1}{z+3} \qquad f_2(z) = \frac{1}{z-3} \qquad f_3(z) = \frac{-1}{z-3}$$



$$-R \longrightarrow -R \qquad -R \longrightarrow R \qquad R \longrightarrow -R \qquad R \longrightarrow R$$



$f_0(-R) = -R$ $f_1(-R) = R$ $f_2(R) = -R$ $f_3(R) = R$



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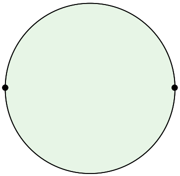
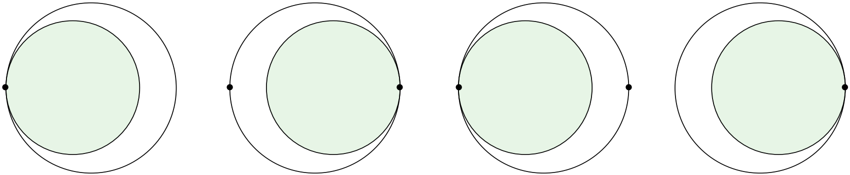
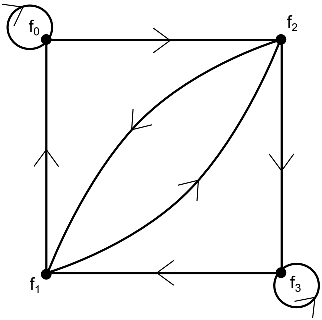
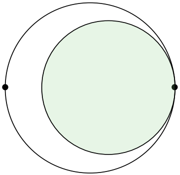


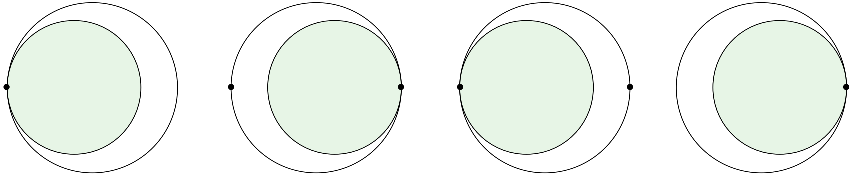


$f_0(-R) = -R$ $\quad f_1(-R) = R$ $\quad f_2(R) = -R$ $\quad f_3(R) = R$



$f_1(D)$





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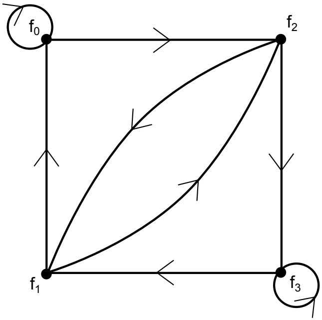
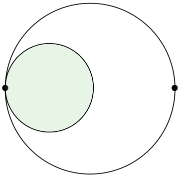
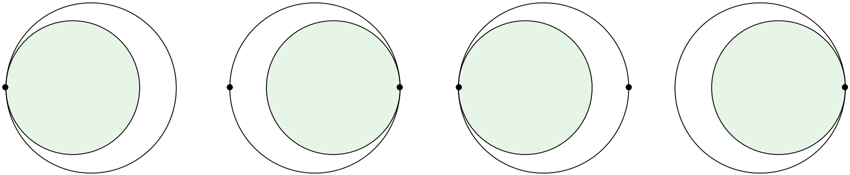


$f_2 \circ f_1(D)$





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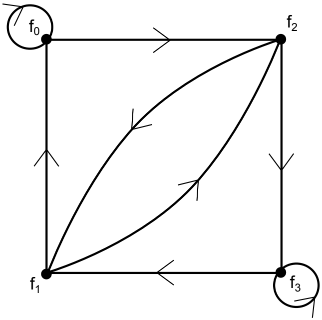
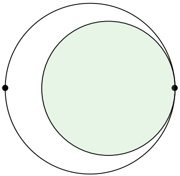
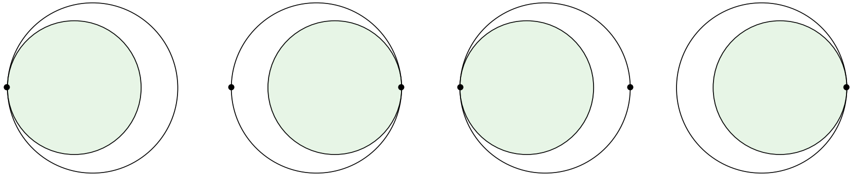


$f_1(D)$





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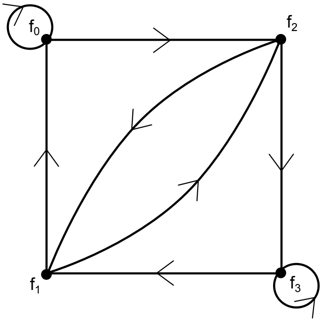
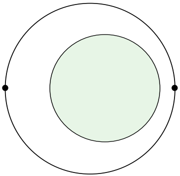


$f_0 \circ f_1(D)$



By construction of G

- there is at most one directed edge between vertices of G, and
- the set

  $$f_{x(1)} \circ f_{x(2)} \circ \cdots \circ f_{x(n)}(D)$$

  is tangent to D if and only if

  $$f_{x(1)} \longrightarrow f_{x(2)} \longrightarrow \; \cdots \longrightarrow f_{x(n)}$$

  is a path in G.

We now load the tangency graph with more information.

To each vertex $f_i \in \mathcal{F}$ we assign a weight $r_i \in \mathbb{R}^+$ as follows:

- If $f_i(D)$ is tangent to D, say $f_i(z_i) \in \partial D$, we set $r_i = 1/|f_i^{'}(z_i)|$,
- otherwise f_i (arbitrarily) set $r_i = 1$.

The theorem which discriminates between limit-point and limit-disc-case can now be restated purely in terms of a tangency graph endowed with vertex weights.

Theorem

Given $\mathcal{F} = \{f_0, \ldots, f_{b-1}\}$*, let* G *be its tangency graph endowed with weighted vertices. The sequence* $x \in \mathcal{S}$ *is limit-disc case if and only if*

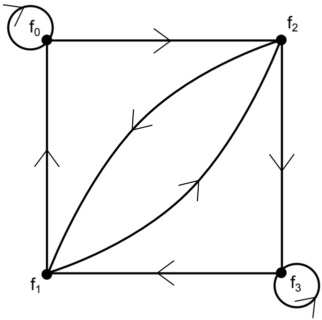
(i) *the vertex sequence*

$$f_{x(1)} \longrightarrow f_{x(2)} \longrightarrow \cdots \longrightarrow f_{x(n)} \longrightarrow \cdots$$

is eventually a path in the tangency graph, and

(ii) $\sum_{n=1}^{\infty} r_{x(1)} \ldots r_{x(n)} < +\infty$.

In our featured example,



each $r_i = \frac{1}{2}(7 - 3\sqrt{5}) < 1$.

Both the graph and its vertex weights restrict which sequences are limit disc type.

If $\mathcal{F}$ induces tangency graph G we can choose $\mathcal{F}^{'}$ which induces the same tangency graph, but with any vertex weights we choose.

However, some directed graphs with at most one directed edge between any two vertices, are not realisable as tangency graphs. This observation invites a more general question, which we do not address:

Question

Given a directed graph with a positive real number assigned to each vertex and at most one directed edge between any two vertices, what is the Hausdorff dimension of the set of vertex sequences which are "limit-disc-case"?

For now suppose the vertex weights are such that the sum

$$\sum_{n=1}^{\infty} r_{x(1)} \cdots r_{x(n)}$$

always converges.

Countable stability of Hausdorff dimension implies

$$\begin{aligned} &\dim(\text{vertex sequences that are paths in } G) \\ = &\dim(\text{vertex sequences that are eventually a path in } G). \\ = &\dim(\mathcal{S}_\circ) \end{aligned}$$

What is the Hausdorff dimension of the subset of $[0,1]$ associated with paths in G?

In the same way that we construct the middle thirds Cantor set, we can remove open intervals of $[0, 1]$ which correspond to finite sequences of maps in $\mathcal{F}$ that are not paths in G.

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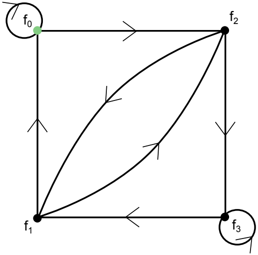
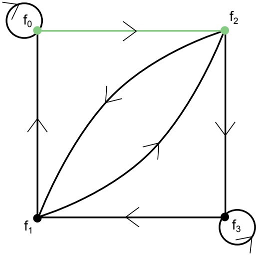
$f_0 \circ \cdots$



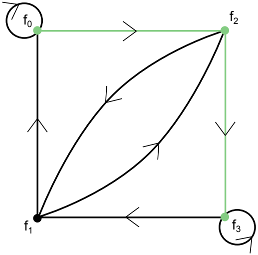
In the same way that we construct the middle thirds Cantor set, we can remove open intervals of $[0,1]$ which correspond to finite sequences of maps in $\mathcal{F}$ that are not paths in G.

$f_0 \circ f_2 \circ \cdots$



In the same way that we construct the middle thirds Cantor set, we can remove open intervals of [0, 1] which correspond to finite sequences of maps in $\mathcal{F}$ that are not paths in G.

$$f_0 \circ f_2 \circ f_3 \circ \cdots$$



For this example, we may compute

$$\dim(\mathcal{S}_\circ) = \frac{\log(2)}{\log(4)}.$$

More generally, we appeal to a 1998 paper[1] by Mauldin and Williams which gives

$$\dim(\mathcal{S}_\circ) = \frac{\log(\rho)}{\log(b)}$$

where ρ is the spectral radius of the graph G, that is the modulus of the largest eigenvalue of its adjacency matrix.

[1] R. D. Mauldin and S. C. Williams, Hausdorff dimension in graph directed constructions, Trans. Amer. Math. Soc. **309** (1988), no. 2, 811–829.

Indeed in our example, G has adjacency matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and spectrum $\{0, 0, 0, 2\}$. Hence $\rho = 2$ and $|\mathcal{F}| = b = 4$ so that

$$\dim(\mathcal{S}_\circ) = \frac{\log(2)}{\log(4)}$$

as expected.

Theorem (J, Short)

Suppose D is a disc and that $\mathcal{F}$ is a finite set of Möbius transformations which each map D into, but not onto itself.
If $\mathcal{S}_\circ$ is the set of sequences in $\mathcal{F}$ that are limit-disc case, then

$$\dim(\mathcal{S}_\circ) \leq \frac{\log(\rho)}{\log(|\mathcal{F}|)}$$

where ρ is the spectral radius of the tangency graph of $\mathcal{F}$.

- For any possible tangency graph, there exists a choice of $\mathcal{F}$ which induces it and attains this bound.
- As a corollary, unless each $f \in \mathcal{F}$ fixes a common point on the boundary of D then $\dim(\mathcal{S}_\circ) < 1$.

Now fix a tangency graph, and ask what values of $\dim(\mathcal{S}_\circ)$ may be attained by varying the vertex weights?

- If each $r_i < 1$ then $\dim(\mathcal{S}_\circ) = \frac{\log(\rho)}{\log(|\mathcal{F}|)}$,
- if each $r_i > 1$ then $\dim(\mathcal{S}_\circ) = 0$.

Are there values in between?

For a general tangency graph the problem is open, but if the graph is complete we have the following:

Proposition (J)

Suppose $\mathcal{F} = \{f_0, \dots, f_{b-1}\}$ has a complete tangency graph with vertex weights $\{r_0, \dots, r_{b-1}\}$.

If $\mathcal{S}_\circ$ is the set of sequences in $\mathcal{F}$ that are limit-disc case, then

$$\dim(\mathcal{S}_\circ) = \frac{\min\left\{\log\left(\sum_{i=0}^{b-1} r_i^{-\nu}\right) \;\middle|\; \nu \geq 0\right\}}{\log(|\mathcal{F}|)}$$

Thank you for your attention!